11/21/19

MIS10 (Continued)

ODE Terminology & Notation

Example of ODE: $y'(t) = 2t$

Q: $y(t) = ?$

A: $y(t) = t^2 + C$ (family of solutions)

Suppose we were also given an initial condition (IC):

$y(0) = 4$

Then the solution to the initial value problem (IVP)

$$\text{IVP} \begin{cases} y'(t) = 2t \\ y(0) = 4 \end{cases}$$

is $y(t) = t^2 + 4$ (one solution)

EM: Euler's method is used to approximate solutions to IVP's.

Idea: (Use Newton's notation for derivative; $y'(t) = \dot{y}(t)$)

$$\dot{y}(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$$

$\Rightarrow$ for small (close to zero) values of h,

$$y(t+h) = y(t) + h \cdot \dot{y}(t) \quad \text{(EM)}$$

step size $= |h|$

If $h > 0$, it's called a forward equation

If $h < 0$, ——— backward equation

Examples:

1) Given $y(0)=3$ and the following table:

t	$\dot{y}(t)$
0	-1
.5	2
1	.5
1.5	-.25
2	-.75

Q: Use EM with a stepsize of 0.5 to approximate $y(2)$

A: (a) Use a forward equation $(h=0.5)$

$$y(t+h) = y(t) + h\cdot\dot{y}(t) \quad \text{EM}$$

$$y(0)=3$$

$t=0$; $h=0.5$ $\quad y(.5) = y(0) + .5\cdot\dot{y}(0) = 3 + .5(-1) = 2.5$

$t=0.5$: $\quad y(1) = y(0.5) + 0.5\cdot\dot{y}(0.5) = 2.5 + 0.5(2) = 3.5$

$t=1$: $\quad y(1.5) = y(1) + 0.5\cdot\dot{y}(1) = 3.5 + 0.5(.5) = 3.75$

$t=1.5$: $\quad y(2) = y(1.5) + 0.5\cdot\dot{y}(1.5) = 3.75 + 0.5(-0.25) = 3.625$

(b) Use a backward equation $(h=-0.5)$

$$y(t+h) = y(t) + h\cdot\dot{y}(t)$$

$t=0.5$: $\quad y(0) = y(.5) - 0.5\cdot\dot{y}(.5) \Rightarrow 3 = y(.5) - 0.5(2) \Rightarrow y(.5) = 4$

$t=1$: $\quad y(0.5) = y(1) - 0.5\cdot\dot{y}(1) \Rightarrow 4 = y(1) - 0.5\cdot(.5) \Rightarrow y(1) = 4.25$

$t=1.5$: $\quad y(1) = y(1.5) - 0.5\cdot\dot{y}(1.5) \Rightarrow 4.25 = y(1.5) - 0.5(-.25) \Rightarrow y(1.5) = 4.125$

$t=2$: $\quad y(1.5) = y(2) - 0.5\cdot\dot{y}(2) \Rightarrow 4.125 = y(2) - 0.5(-.75) \Rightarrow y(2) = 3.75$

For us, we have $y(t) = {}_tP_x^{ij}$

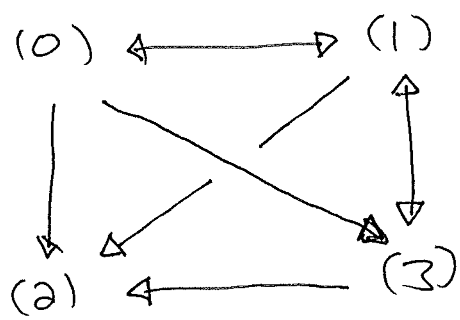
Q: How do we get ${}_t\dot{P}_x^{ij}$ (derivatives)

A: ${}_t\dot{P}_x^{ij}$ = "rate in" — "rate out"

"rate in" = rate of instantaneous transition in to state j at time t for someone starting in state i

"rate out" = rate of instantaneous transition out of state j at time t for someone starting in state i

Example:



Determine ${}_t\dot{P}_x^{01}$ (= [rate in] − (rate out))

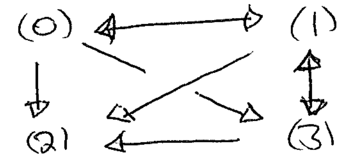
$$\text{rate in} = {}_tP_x^{00} \cdot \mu_{x+t}^{01} + {}_tP_x^{03} \cdot \mu_{x+t}^{31}$$

$$\text{rate out} = {}_tP_x^{01} \cdot \mu_{x+t}^{10} + {}_tP_x^{01} \cdot \mu_{x+t}^{12} + {}_tP_x^{01} \cdot \mu_{x+t}^{13}$$

$$= {}_tP_x^{01} \cdot \left(\mu_{x+t}^{10} + \mu_{x+t}^{12} + \mu_{x+t}^{13} \right) = {}_tP_x^{01} \cdot \mu_{x+t}^{1\bullet}$$

$$\therefore \; {}_t\dot{P}_x^{01} = \left[{}_tP_x^{00} \cdot \mu_{x+t}^{01} + {}_tP_x^{03} \cdot \mu_{x+t}^{31} \right] - \left({}_tP_x^{01} \cdot \mu_{x+t}^{1\bullet} \right)$$

Now determine ${}_t\mathring{P}_x^{30}$.



rate in: there's only one "instantaneous" way into state 0, and that's from state 1

$$\therefore \text{rate in} = {}_tP_x^{31} \cdot \mu_{x+t}^{10}$$

Note that the path (3) → (1) → (0) is <u>not</u> instantaneous since it takes two transitions.

rate out: there are three ~~three~~ "instantaneous" ways out of state 0; to state (1) or (2) or (3).

$$\therefore \text{rate out} = {}_tP_x^{30} \cdot \mu_{x+t}^{01} + {}_tP_x^{30} \cdot \mu_{x+t}^{02} + {}_tP_x^{30} \cdot \mu_{x+t}^{03}$$

$$= {}_tP_x^{30} \cdot \left(\mu_{x+t}^{01} + \mu_{x+t}^{02} + \mu_{x+t}^{03}\right)$$

$$= {}_tP_x^{30} \cdot \mu_{x+t}^{0\tau}$$

$$\therefore \quad {}_t\mathring{P}_x^{30} = [\text{rate in}] - (\text{rate out})$$

$$= \left[{}_tP_x^{31} \cdot \mu_{x+t}^{10}\right] - \left({}_tP_x^{30} \cdot \mu_{x+t}^{0\tau}\right)$$